Electromagnetic Fields

Part 2

Lec 01

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Chapter Contents

2.4 Maxwell's First Equation (Continued)

Steady or Static Electric (Electrostatic) Field

- Eqn. (2.29a) is known as ***Gauss's law*** which simply means that the total electric flux through any closed surface is equal to the charge enclosed.
- This equation is also ***one of Maxwell's four equations*** in integral form, as they apply to electrostatic and electromagnetic fields.
- The differential form of ***Gauss's or Maxwell's equation*** can be derived by applying the divergence theorem Eqn. (1.35) as follows:

$$\oint_S \vec{D}_S . d\vec{s} = Q$$

divide by Δv gives:

$$\frac{\oint_S \vec{D}_S . d\vec{s}}{\Delta v} = \frac{Q}{\Delta v} \qquad (2.31a)$$

- Take the limit as the volume shrinks to zero, use (1.35):
- Therefore, from Eqn. (2.25a) we have:

$$Div\vec{A} = \nabla.\vec{A} = \lim_{\Delta v \to 0} \frac{\oint_s \vec{A}.d\vec{S}}{\Delta v}$$

$$\nabla.\vec{D} = \rho_v \qquad (2.31b)$$

- ***Conclusion: Eqns. (2.31a) and (2.31b) represent Gauss's law or one of the four Maxwell's equation in both integral and differential form.***

$$\psi = \oint_S \vec{D}_S . d\vec{s} = Q = \int_{vol} \rho_v dv$$

Intgral Form

$$\nabla.\vec{D} = \rho_v$$

Diffentional Form

This is known as Gauss's Law for Electrostatic Field

2.5 Electric Potential

- As far as charges are concerned, electric field is a force field and hence work is associated with the movement of charges in an electric field. If a test charge is moved along the direction of the field, work done by the field and hence it accelerates the test charge. If the charge is moved against the direction of the field, an external agent has to supply the energy to overcome the force exerted on the charge by field.
- Let us consider the displacement of a test charge 'Q' by an infinitesimal distance 'dl' from A to B at an angle α with the electric field $\vec{E}$ at the point 'A' as shown in Fig. 2.17.
- The force exerted on the test charge by the field has magnitude Q$\vec{E}$ and is directed along $\vec{E}$. Its component along the line from A to B is Q$\vec{E}$cos α. If the charge is moved from A to B, the amount of work, dW done by the field is the product of the force and the displacement, that is:

$$dW = Q\,|\vec{E}|\cos\alpha\,|\vec{dl}| = Q\vec{E}.\vec{dl} \qquad (2.32)$$

$\vec{dl}$ is the vector from A to B

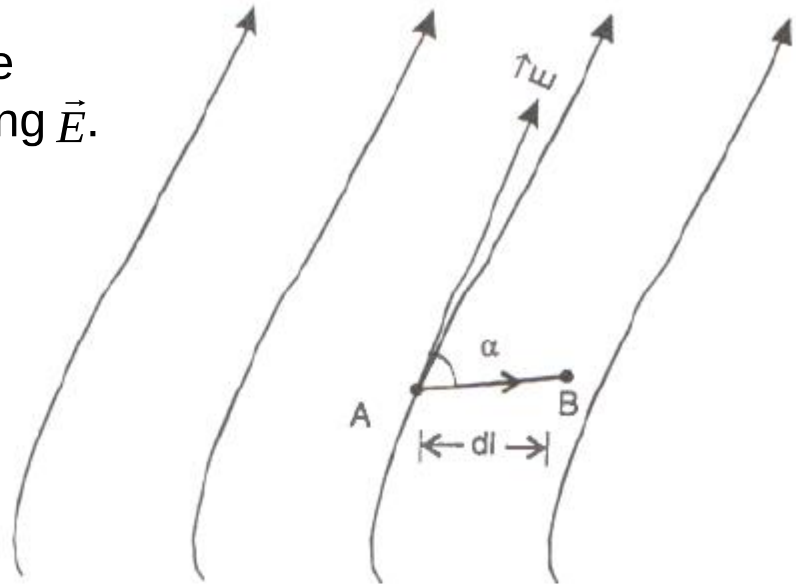

Fig. 2.17 Movement of charge in an electric field

2.5 Electric Potential (Continued)

- Note that dW is positive if $0 < \alpha < 90^0$, so that work is done by the field; dW is negative if $90^0 < \alpha < 180°$, so that negative work is done by the field, that is, work is done by an external agent against the field.
- Thus the total work done (or) the potential energy required in moving Q from A to B against the field is:

$$W = -Q\int_A^B \vec{E}.d\vec{l} \qquad (2.33)$$

- When W is divided by Q, it gives the potential energy per unit charge. This quantity is denoted by VAB and is known as potential difference between A and B.

$$V_{AB} = \frac{W}{Q} = -\int_A^B \vec{E}.d\vec{l} \qquad (2.34)$$

- V_{AB} is measured in joule per Coulomb, commonly referred to as Volt.

2.5 Electric Potential (Continued)

2.5.1 The Potential Field of Point Charges

- Consider two point charges A and B in the electric field of a point charge Q situated at distances r_A and r_B respectively, from the point charge as shown in Fig. 2.18. Using the eqns. (2.9) and (2.34), the potential difference V_{AB} can be computed for any specified path from A to B as:

$$\vec{E} = \frac{Q}{4\pi\varepsilon_o r^2}\hat{a}_r \ V/m$$

and $d\vec{l}$ in spherical co-ordinates (1.10):

$$d\vec{l} = dr\,\hat{a}_r + rd\theta\,\hat{a}_\theta + r\sin\theta d\phi\,\hat{a}_\phi$$

Then: $$V_{AB} = -\int_A^B \vec{E}.d\vec{l}$$

$$V_{AB} = -\int_A^B \frac{Q}{4\pi\varepsilon_o r^2}\vec{a}_r.(dr\hat{a}_r + rd\theta\hat{a}_\theta + r\sin\theta d\phi\hat{a}_\theta)$$

$$V_{AB} = -\int_A^B \frac{Q}{4\pi\varepsilon_o r^2}dr \quad (2.35)$$

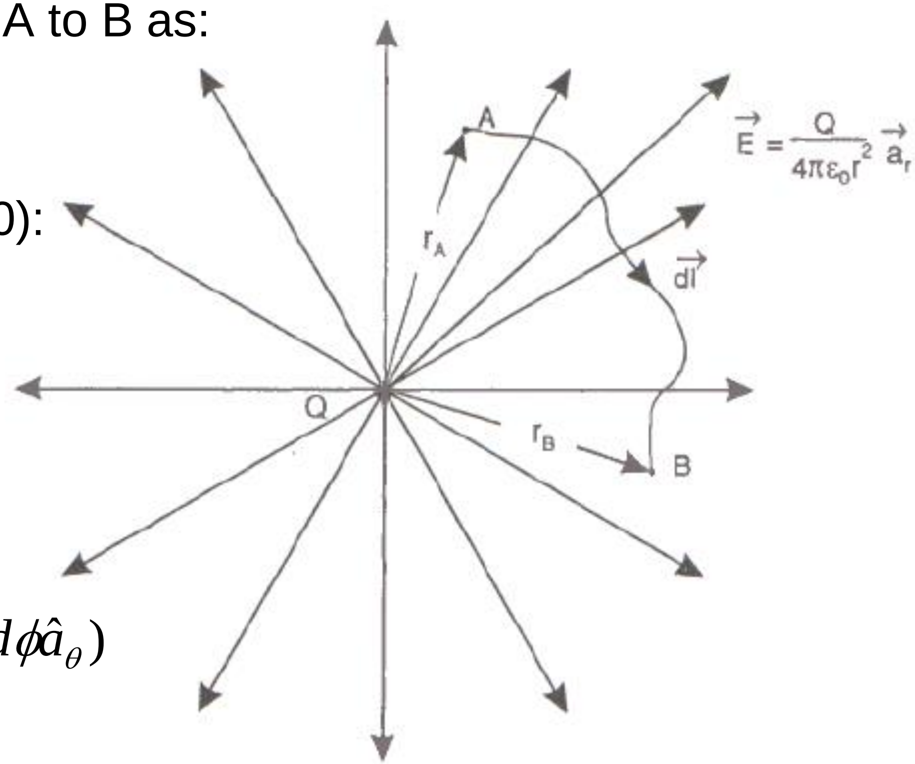

Fig. 2.18 Potential difference between two points in the electric field of point charge.

2.5 Electric Potential (Continued)

2.5.1 The Potential Field of Point Charges

$$V_{AB} = -\int_{A}^{B} \frac{Q}{4\pi\varepsilon_o r^2} dr = \frac{Q}{4\pi\varepsilon_o r_b} - \frac{Q}{4\pi\varepsilon_o r_a} \qquad (2.36a)$$

Where $V_{AB} = V_B - V_A \qquad V_B = \frac{Q}{4\pi\varepsilon_o r_b} \;\; and \; V_A = \frac{Q}{4\pi\varepsilon_o r_a}$ (2.36b)

- From eqn. (2.36), it is known that, for a given charge Q, the potential difference between two points is dependent only upon their distances from the point charge and not on the path from A to B chosen for its evaluation.
- The potential at any point is the potential difference between that point and an arbitrary reference point at which the potential is zero. In case of a point charge, a convenient reference point is $r_o = \infty$. So:

$$V(r) = \frac{Q}{4\pi\varepsilon_o r} \qquad (2.37a)$$

- The potential at a distance, r from the point charge is thus the work done per unit charge by the external agent in bringing a test charge from infinity to that point, that is:

$$V(r) = -\int_{\infty}^{r} \vec{E}.d\vec{l} \qquad (2.37b)$$

2.5 Electric Potential (Continued)

2.5.1 The Potential Field of Point Charges

Example 2.11:

(a) What is the absolute potential at a point P which is 2 m from a point charge Q = 5 μC.

(b) What is the work required to move a 8 nC charge from ∞ to P?

Solution:

(a) From (2.37), potential at a point P is given by:

$$V = \frac{Q}{4\pi\varepsilon_o r} = \frac{5x10^{-6}}{4\pi x 8.85x10^{-12}\,x2} = 22.5\ kV$$

(b) From (2.28), the required work, W is given by:

$$V_A = \frac{W}{Q} \quad \Rightarrow \quad W = QxV$$

$$= 8x10^{-9}\,x22.5x10^{3} = 180x10^{-6} = 180\ \mu J$$

2.5 Electric Potential (Continued)

2.5.2 Potential Gradient

- The potential difference between points A and B is independent of the path taken. Hence, we get:

$$V_{AB} = -V_{BA}$$

- From the Fig. 2.19,

$$V_{AB} + V_{BA} = \oint_L \vec{E}.d\vec{l} = 0 \quad \Rightarrow \quad \oint_L \vec{E}.d\vec{l} = 0 \qquad (2.38a)$$

- This shows that the line integral of $\vec{E}$ along a closed path must be zero. That is, no work is done in moving a charge along a closed path in an electrostatic field.
- Applying Stoke's theorem (1.39), then (2.38) becomes:

$$\oint_L \vec{E}.d\vec{l} = \int_S (\nabla X \vec{E}).d\vec{S} = 0 \quad \Rightarrow \quad \nabla X \vec{E} = 0 \quad (2.38b)$$

- Any vector field that satisfies either of two equations, eqn.(2.38) is said to be conservative field (or) irrotational field. Thus an electrostatic field is a conservative field.

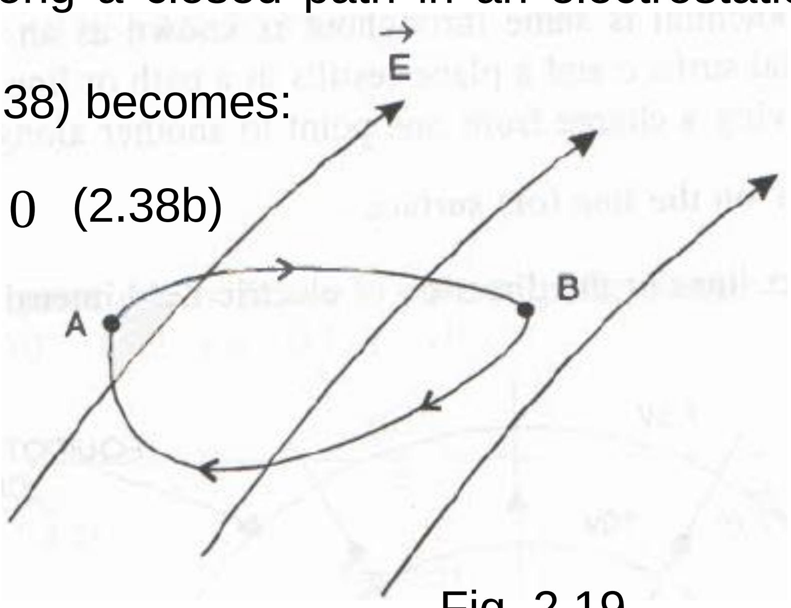

Fig. 2.19

2.5 Electric Potential (Continued)

2.5.2 Potential Gradient

- Eqn.(2.38) is called Faraday's law and also ***one of Maxwell's four equations.***

$$\oint_L \vec{E}.d\vec{l} = \int_S (\nabla X \vec{E}).d\vec{S} = 0 \qquad \nabla X \vec{E} = 0 \qquad (2.38)$$

Intgral Form *Diffentional Form*

This is known as Faraday's Law for Electrostatic Field

- From (2.37), we have: $V = -\int_L \vec{E}.d\vec{l}$
- Differentiate on both sides results in: $dV = -\vec{E}.d\vec{l}$

$$-\vec{E}.d\vec{l} = -(E_x\hat{a}_x + E_y\hat{a}_y + E_z\hat{a}_z).(dx\hat{a}_x + dy\hat{a}_y + dz\hat{a}_z) = E_x dx + E_y dy + E_z dz$$

- But dV can be written as: $dV = \frac{\partial V}{\partial x}dx + \frac{\partial V}{\partial y}dy + \frac{\partial V}{\partial z}dz$
- Comparing the above two equations, we conclude:

$$E_x = \frac{\partial V}{\partial x} \;,\; E_y = \frac{\partial V}{\partial y} \;\; and \;\; E_z = \frac{\partial V}{\partial z} \quad \Rightarrow \quad \vec{E} = -\nabla V \qquad (2.39)$$

- From (2.39), $\vec{E}$ is the gradient of V (or) potential gradient. The negative sign shows that the direction of E is opposite to the direction in which V increases.

2.5 Electric Potential (Continued)

2.5.2 Potential Gradient

Example 2.12:

The electric potential near the origin of a system of coordinates is given by:

$$V = 2x^2 + 3y^2 + 5z^2$$

Find the electric field at (l, 2, 3).

Solution:

From (2.30), $\vec{E}$ is given by:

$$\vec{E} = -\nabla V = -\left(\frac{\partial V}{\partial x}\hat{a}_x + \frac{\partial V}{\partial y}\hat{a}_y + \frac{\partial V}{\partial z}\hat{a}_z\right)$$

$$\vec{E} = -(4x\hat{a}_x + 6y\hat{a}_y + 10z\hat{a}_z)$$

At point (l, 2, 3), $\vec{E}$ is given by:

$$\vec{E}(1,2,3) = -(4\hat{a}_x + 12\hat{a}_y + 30\hat{a}_z)$$

2.5 Electric Potential (Continued)

2.5.3 Equipotential Surface

- Any surface on which the potential is same throughout is known as an equipotential surface. The intersection of an equipotential surface and a plane results in a path or line known as an equipotential line. No work is done in moving a charge from one point to another along an equipotential line (or) surface and hence:

$$\oint_L \vec{E}.d\vec{l} = 0 \quad \text{on the line (or) surface.} \quad (2.34)$$

- The lines of force or flux lines or the direction of $\vec{E}$ is always normal to equipotential surfaces as shown in Fig. 2.20.

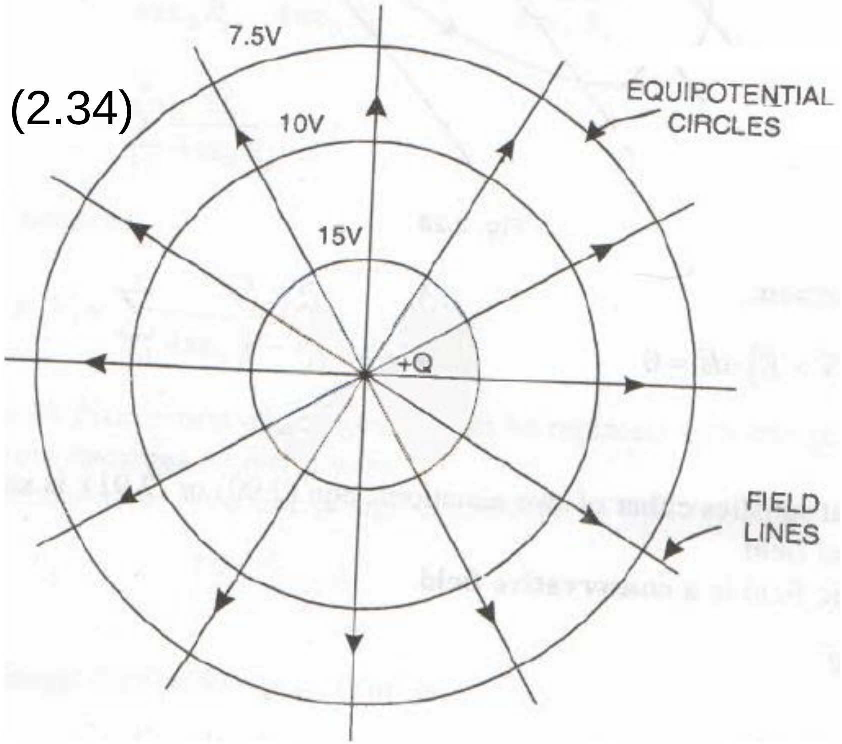

Fig. 2.20 Equipotential lines of a point charge

2.5 Maxwell's Second Equation

Steady or Static Electric (Electrostatic) Field

- Eqn.(2.38) is called Faraday's law and also ***one of Maxwell's four equations.***

$$\oint_L \vec{E}.d\vec{l} = \int_S (\nabla X \vec{E}).d\vec{S} = 0$$

Intgral Form

This is known as Faraday's Law for Electrostatic Field

(2.38)

$$\nabla X \vec{E} = 0$$

Diffentional Form

ex1 : Find the work required to move 2C charge from B to A in the Same field $E = y\bar{a}_x + x\bar{a}_y + 2\bar{a}_z$

B (1,0,1)   A(0.8, 0.6, 1) along the shorter arc of the circle $x^2+y^2=1$ $z=1$

Solution

$$\because W = -\int_B^A Q\bar{E}\cdot d\bar{L}$$

$$= -2\int_B^A (y\bar{a}_x + x\bar{a}_y + 2\bar{a}_z)\cdot(dx\,\bar{a}_x + dy\,\bar{a}_y + dz\,\bar{a}_z)$$

$$= -2\left[\int_{B\,1}^{A\,0.8} y\,dx + \int_{B\,0}^{A\,0.6} x\,dy + \int_{B\,1}^{A\,1} 2\,dz\right]$$

⇒ from the circle equ. we find a relation between x & y

$$\therefore W = -2\left[\int_1^{0.8}\sqrt{1-x^2}\,dx + \int_0^{0.6}\sqrt{1-y^2}\,dy + \int_1^1 2\,dz\right]$$

$$= \boxed{-0.96\ J}$$

EX2 Use the same data in ex1 to get the work done to move the 2C charge from B→A through the straight line path between A & B.

$$\frac{y-0}{x-1} = \frac{0.6}{0.8-1} \Rightarrow \frac{y}{x-1} = \frac{0.6}{-0.2} \Rightarrow$$

$$y = -(3x-3) \longrightarrow (1)$$

$$\frac{z-1}{y-0} = 0 \Rightarrow z = 1 \longrightarrow (2)$$

$$\therefore W = -2\int_1^{0.8} y\,dx - 2\int_0^{0.6} x\,dy - 4\int_1^1 dz$$

$$= -2\int_1^{0.8}(-3x+3)\,dx - 2\int_0^{0.6} \frac{3-y}{3}\,dy - 0$$

$$= -2\left[-3\frac{x^2}{2}+3x\right|_1^{0.8} - 2\left[y - \frac{1}{3}\frac{y^2}{2}\right|_0^{0.6}$$

$$= -2\left[\left(-\frac{3}{2}\frac{0.8^2}{2}+2.4\right) - \left(-3\frac{1}{2}+3\right)\right]$$

$$-2\left[0.6 - \frac{1}{3}\frac{0.6^2}{2}\right]$$

$$= -0.96\,J$$

Ex4

Calculate the work done in moving 4-C charge from B(1,0,0) to A(0,2,0) along the path $y = 2-2x$, $z=0$ in the field

(a) $E = 5\bar{a}_x$ (b) $5x a_x$ (c) $5x\bar{a}_x + 5y\bar{a}_y$

Solution

$$dL = dx\,\bar{a}_x + dy\,\bar{a}_y + dz\,\bar{a}_z \longrightarrow (1)$$

$$W = -Q\int_B^A E\cdot dL =$$

(a)
$$= -4\left[\int_B^A 5\bar{a}_x\cdot(dx\,\bar{a}_x + dy\,\bar{a}_y + dz\,\bar{a}_z)\right)$$

$$= -4\left[\int_1^0 5dx\right] = -4\left[5x\Big|_1^0\right.$$

$$= -4[-5] = \boxed{20}\,J$$

(b)
$$W = -4\left[5\frac{x^2}{2}\Big|_1^0\right. = -\frac{20}{2}[-1] = \boxed{10J}$$

(c)
$$W = -4\left[\int_1^0 5x\,dx + \int_0^2 5y\,dy\right]$$

$$= 10 + \left(-4 * 5\frac{y^2}{2}\Big|_0^2\right) = 10 - (+10*4)$$

$$= \boxed{-30J}$$

$$V = -\int_L \vec{E}.d\vec{l}$$

$$\boxed{E = -\nabla V}$$

∇ → grad

$$\nabla V = \frac{\partial V}{\partial x}\bar{a}_x + \frac{\partial V}{\partial y}a_y + \frac{\partial V}{\partial z}$$

Rectangular

$$\nabla V = \frac{\partial V}{\partial \rho}\bar{a}_\rho + \frac{1}{\rho}\frac{\partial V}{\partial \phi}\bar{a}_\phi + \frac{\partial V}{\partial z}a_z$$

Cylindrical

$$\nabla V = \frac{\partial V}{\partial r}\bar{a}_r + \frac{1}{r}\frac{\partial V}{\partial \theta}\bar{a}_\theta + \frac{1}{r\sin\theta}\frac{\partial V}{\partial \phi}a_\phi$$

spherical

EX If $E = 6x^2\bar{a}_x + 6y\bar{a}_y + 4\bar{a}_z$ V/m

Find (a) V_{MN} $M(2,6,-1)$ $N(-3,-3,2)$

$$V_{MN} = -\int E \cdot dL$$

$$= -\int (6x^2 a_x + 6y\bar{a}_y + 4\bar{a}_z) \cdot (dx\,\bar{a}_x + dy\,\bar{a}_y + dz\,\bar{a}_z)$$

$$= -\left[\int_{-3}^{2} 6x^2 dx + \int_{-3}^{6} 6y\,dy + \int_{2}^{-1} 4\,dz\right]$$

$$= -\left(\frac{6x^3}{3}\Big|_{-3}^{2} + \frac{6y^2}{2}\Big|_{-3}^{6} + 4z\Big|_{2}^{-1}\right)$$

$$= -[2(8-(-27)) + 3(36-9) + (4(-1-2))]$$

$$= -[2(35) + 3*27 - 4*3]$$

$$= -[70 + 71 - 12]$$

$$= -[141 - 12]$$

$$\boxed{V_{MN} = -139}$$

(b) V_M if $V = 0$ at $Q(4,-2,-35)$

(c) V_N if $V = 2$ at $P(1,2,-4)$

Thank you for your attention

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